L11 Tests with upper-sided alternative hypotheses

1. Tests with monotone increasing likelihood ratio in $T(X)$.

   (1) Definition of Monotone increasing likelihood ratio in $T(X)$
   A population system has parameter θ. With sample X the likelihood function is $L(\theta) = f(X;\theta)$ where $f(X;\theta)$ is the joint pdf or pmf of X. Suppose $T(X)$ is a statistic. The system has monotone increasing likelihood ratio in $T(X)$ if $\Lambda = \frac{L(\theta_2)}{L(\theta_1)}$ is an increasing function of $T(X)$ for all $\theta_1 < \theta_2$.

   (2) Invariant critical function for LRT
   For $H_0 : \theta = \theta_1$ versus $H_a : \theta = \theta_2$ where $\theta_1 < \theta_2$
   the LRT has critical function

$$\phi(X) = \begin{cases} 1 & \Lambda = \frac{L(\theta_2)}{L(\theta_1)} > k \\ r & \Lambda = \frac{L(\theta_2)}{L(\theta_1)} = k \\ 0 & \Lambda = \frac{L(\theta_2)}{L(\theta_1)} < k \end{cases} = \begin{cases} 1 & T(X) > c \\ r & T(X) = c \\ 0 & T(X) < c \end{cases}$$

   This $\phi(X)$ is invariate with θ_1 and θ_2 as long as $\theta_1 < \theta_2$, i.e.,
   for $H_0 : \theta = \theta_3$ versus $H_a : \theta = \theta_4$ where $\theta_3 < \theta_4$ the LRT has the same general form of critical function since $T(X)$ does not depend on any specific $\theta_1 < \theta_2$.

   (3) Property 1 of $E_\theta[\phi(X)]$
   If $\psi(X)$ is also a critical function, i.e., $0 \le \psi(X) \le 1$, and $E_{\theta_0}[\psi(X)] \le E_{\theta_0}[\phi(X)]$, then $E_\theta[\psi(X)] \le E_\theta[\psi(X)]$ for all $\theta > \theta_0$.

   Proof. Let $\theta_1 > \theta_0$. We show $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.
   LRT on $H_0 : \theta = \theta_0$ versus $\theta = \theta_1$ where $\theta_1 > \theta_0$ has critical function in (2).
   Denote $E_{\theta_0}[\phi(X)]$ as α. Then $\phi(X)$ is α-level MP test.
   Thus if $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\phi(X)]$, then $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.

   (4) Property 2 of $E_\theta[\phi(X)]$
   $E_\theta[\phi(X)]$ is a non-decreasing function of θ.

   Proof. Let $\theta_1 < \theta_2$. We show $E_{\theta_1}[\phi(X)] \le E_{\theta_2}[\phi(X)]$.
   LRT test in (2) has critical function $\phi(X)$. Denote $E_{\theta_1}[\phi(X)]$ as α.
   Then $\phi(X)$ gives α-level MP test. By HW, this test is unbiased.
   So $E_{\theta_1}[\phi(X)] = \alpha \le E_{\theta_2}[\phi(X)]$.

2. α-level uniformly most powerfu test

   (1) Definition For $H_0 : \theta \in H_0$ versus $H_a : \theta \in H_a$, $\phi(X)$ is α-level uniformly most powerful (UMP) test if
   (i) $E_\theta[\phi(X)] \le \alpha$ for all $\theta \in H_0$, i.e., $\phi(X)$ is an α-level test.
   (ii) For all α-level test $\psi(X)$, $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta \in H_a$ (UMP)

   (2) α-level UMP on $H_0 : \theta = \theta_0$ versus $H_a : \theta > \theta_0$
   For above hypotheses, $\phi(X)$ in (2) of 1 with $E_{\theta_0}[\phi(X)] = \alpha$ is α-level UMP test.

   Proof. $E_{\theta_0}[\phi(X)] = \alpha$. So (i) in (1) of 2 is satisfied.
   Suppose $\psi(X)$ is also an α-level test. Then $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\phi(X)]$.
   By the Property 1 for $E_\theta[\phi(X)]$ in (3) of 1, $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta > \theta_0$.
   Hence (ii) of 2 is met. Therefore $\phi(X)$ is an α-level UMP test.

3. Examples

Ex1: Population $N(\mu, \sigma^2)$ with known σ^2 has likelihood function

$$L(\mu) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{\sum X_i^2 - 2\mu \sum X_i + n\mu^2}{-2\sigma^2}\right).$$

With $\mu_1 < \mu_2$, $\Lambda = \frac{L(\mu_2)}{L(\mu_1)} = \frac{\exp\left(\frac{-2\mu_2 \sum X_i + n\mu_2^2}{-2\sigma^2}\right)}{\exp\left(\frac{-2\mu_1 \sum X_i + n\mu_1^2}{-2\sigma^2}\right)} = \exp\left[\frac{2(\mu_2-\mu_1)\overline{X}_n + n(\mu_1^2 - \mu_2^2)}{2\sigma^2}\right]$ is an increasing function of $\overline{X}_n$. Thus the system has monotone increasing likelihood ratio in $\overline{X}_n$.

Ex2: In the setting of Ex1, note that $\overline{X}_n$ is an increasing function of $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$. Thus the system also has monotone increasing likelihood function in $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$.

Ex3: With $\phi(X) = \begin{cases} 1 & \overline{X}_n > c_1 \\ 0 & \overline{X}_n \le c_1 \end{cases}$, solving

$$\alpha = E_{\mu_0}[\phi(X)] = P_{\mu_0}(\overline{X}_n > c_1) = P\left(N\left(0, \frac{\sigma^2}{n}\right) > c_1\right) = P\left(N(0, 1^2)) > \frac{c_1 - \mu_0}{\sigma/\sqrt{n}}\right),$$

We have $\frac{c_1 - \mu_0}{\sigma/\sqrt{n}} = Z_\alpha \Longrightarrow c_1 = \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}}$. Therefore

> $H_0 : \mu = \mu_0$ versus $H_a : \mu > \mu_0$
> Critical function: $\phi(X) = \begin{cases} 1 & \overline{X}_n > \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \\ 0 & \overline{X}_n \le \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}} \end{cases}$

gives an α-level UMP test

Comment 1: The test schem im Ex3 can be expreesed via test statistics and rejection rule:

> $H_0 : \mu = \mu_0$ versus $H_a : \mu > \mu_0$
> Test statistic: $\overline{X}_n = \frac{\sum X_i}{n}$
> Rejet H_0 if $\overline{X}_n > \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}}$

gives an α-level UMP test

Comment 2: One can also define $\phi(X)$ using $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$ or use Z as a traditional test statistic.

L12 Tests with lower-sided alternative hypotheses

1. Tests with monotone increasing likelihood ratio in $T(X)$.

   (1) Invariant critical function
   Suppose a system has monotone increasing likelihood ratio in $T(X)$, i.e., $\Lambda = \frac{L(\theta_2)}{L(\theta_1)}$ is an increasing function of $T(X)$ for all $\theta_1 < \theta_2$.
   For $H_0 : \theta = \theta_2$ versus $H_a : \theta = \theta_1$ where $\theta_1 < \theta_2$
   the LRT has critical function

$$\phi(X) = \begin{cases} 1 & \frac{L(\theta_1)}{L(\theta_2)} > k \\ r & \frac{L(\theta_1)}{L(\theta_2)} = k \\ 0 & \frac{L(\theta_1)}{L(\theta_2)} < k \end{cases} = \begin{cases} 1 & \Lambda < \frac{1}{k} \\ r & \Lambda = \frac{1}{k} \\ 0 & \Lambda > \frac{1}{k} \end{cases} = \begin{cases} 1 & T(X) < c \\ r & T(X) = c \\ 0 & T(X) > c \end{cases}$$

   This $\phi(X)$ is invariate with θ_1 and θ_2 as long as $\theta_1 < \theta_2$, i.e.,
   for $H_0 : \theta = \theta_4$ versus $H_a : \theta = \theta_3$ where $\theta_3 < \theta_4$ the LRT has the same general form of critical function since $T(X)$ does not depend on any specific $\theta_1 < \theta_2$.

   (2) Property 1 of $E_\theta[\phi(X)]$
   If $\psi(X)$ is also a critical function, i.e., $0 \le \psi(X) \le 1$, and $E_{\theta_0}[\psi(X)] \le E_{\theta_0}[\phi(X)]$, then $E_\theta[\psi(X)] \le E_\theta[\psi(X)]$ for all $\theta < \theta_0$.

   Proof. Let $\theta_1 < \theta_0$. We show $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.
   LRT on $H_0 : \theta = \theta_0$ versus $\theta = \theta_1$ where $\theta_1 < \theta_0$ has critical function in (1).
   Denote $E_{\theta_0}[\phi(X)]$ as α. Then $\phi(X)$ is α-level MP test.
   Thus if $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\phi(X)]$, then $E_{\theta_1}[\psi(X)] \le E_{\theta_1}[\phi(X)]$.

   (3) Property 2 of $E_\theta[\phi(X)]$
   $E_\theta[\phi(X)]$ is a non-increasing function of θ.

   Proof. Let $\theta_1 < \theta_2$. We show $E_{\theta_1}[\phi(X)] \ge E_{\theta_2}[\phi(X)]$.
   LRT test in (1) has critical function $\phi(X)$. Denote $E_{\theta_2}[\phi(X)]$ as α.
   Then $\phi(X)$ gives α-level MP test. By HW, this test is unbiased.
   So $E_{\theta_1}[\phi(X)] \ge \alpha = E_{\theta_2}[\phi(X)]$.

2. α-level uniformly most powerfu test

   (1) α-level UMP on $H_0 : \theta = \theta_0$ versus $H_a : \theta < \theta_0$
   For above hypotheses, $\phi(X)$ in (1) of 1 with $E_{\theta_0}[\phi(X)] = \alpha$ is α-level UMP test.

   Proof. $E_{\theta_0}[\phi(X)] = \alpha$. So $\phi(X)$ gives an α-level test.
   For all α-level test $\psi(X)$, $E_{\theta_0}[\psi(X)] \le \alpha = E_{\theta_0}[\phi(X)]$.
   By the Property 1, $E_\theta[\psi(X)] \le E_\theta[\phi(X)]$ for all $\theta < \theta_0$.
   Hence $\phi(X)$ is an α-level UMP test.

   (2) Summary
   Suppose likelihood ratio is monotone increasing function of $T(X)$. Then

   For $H_0 : \theta = \theta_0$ versus $H_a : \theta < \theta_0$

$$\phi(X) = \begin{cases} 1 & T(X) < c \\ r & T(X) = c \\ 0 & T(X) > c \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha$$

   is an α-level UMP test

For $H_0: \theta = \theta_0$ versus $H_a: \theta > \theta_0$
$$\phi(X) = \begin{cases} 1 & T(X) > c \\ r & T(X) = c \\ 0 & T(X) < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha$$
is an α-level UMP test

3. Examples

Ex1: $N(\mu, \sigma^2)$ has monotone likelihood ratio in $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$.

For $H_0: \mu = \mu_0$ versus $H_a: \mu < \mu_0$, $\phi(X) = \begin{cases} 1 & Z < c \\ 0 & Z \geq c \end{cases}$ where

$$\alpha = E_{\mu_0}[\phi(X)] = P_{\mu_0}(Z(X) < c) = P\left(N(0, 1^2) < c\right) \Longrightarrow c = -Z_\alpha.$$

Thus $\phi(X) = \begin{cases} 1 & Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}} < -Z_\alpha \\ 0 & Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}} \geq -Z_\alpha \end{cases}$ gives α-level UMP test.

This test can be expressed using test statistic and rejection rule.

$H_0: \mu = \mu_0$ versus $H_a: \mu < \mu_0$
Test statistic: $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$
Reject H_0 if $Z < -Z_\alpha$

gives an α-level UMP test

Comment 1: The test schem im Ex3 can be expreesed via test statistics and rejection rule:

$H_0: \mu = \mu_0$ versus $H_a: \mu > \mu_0$
Test statistic: $\overline{X}_n = \frac{\sum X_i}{n}$
Rejet H_0 if $\overline{X}_n > \mu_0 + Z_\alpha \frac{\sigma}{\sqrt{n}}$

gives an α-level UMP test

Comment 2: One can also define $\phi(X)$ using $Z = \frac{\overline{X}_n - \mu_0}{\sigma/\sqrt{n}}$ or use Z as a traditional test statistic.